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"SOFT" CAPTURE IN PONTRYAGIN'S EXAMPLE WITH MANY PARTICIPANTS†

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The conditions of "soft" capture in Pontryagin's example with many participants and equal resources of the players are obtained.

Pshenichnyi [1] obtained the necessary and sufficient conditions for the capture by a group of pursuers of a single evader in the problem of simple pursuit with equal resources of the players. The only extension of the given problem is Pontryagin's example [2] with many participants and identical dynamic and inertial resources of the players. The problem has already been examined [3] for the case where matching of the phase coordinates is a condition of capture. In the present paper, matching not only of the phase coordinates but also of the velocities is a condition of capture. On the assumption that the roots of the characteristic equation are real and non-positive, sufficient conditions of capture are obtained in terms of the initial positions. For inertial objects, the conditions are obtained for the capture by a group of pursuers of at least one evader, provided that all evaders use the same control. The present paper touches on earlier investigations [2–6].

1. THE CAPTURE OF A SINGLE EVADER

In the space R^k $(k \geq 2)$, we consider a differential game Γ of $n+1$ persons: n pursuers $P_1, P_2, \ldots, P_n$ and an evader E. The law of motion of each of the pursuers P_i has the form

$$x_i^{(l)} + a_1 x_i^{(l-1)} + \ldots + a_l x_i = u_i, \quad u_i \in V \tag{1.1}$$

The law of motion of the evader E has the form

$$y^{(l)} + a_1 y^{(l-1)} + \ldots + a_l y = \upsilon, \quad \upsilon \in V \tag{1.2}$$

Here, x_i, y_j, u_i and $\upsilon \in R^k$, $a_1, \ldots, a_l \in R^1$, and V is a compactum. When $t = 0$, the initial conditions

$$x_i^{(\alpha)}(0) = x_{i\alpha}^0, \quad y^{(\alpha)}(0) = y_\alpha^0, \quad \alpha = 0, \ldots, l-1$$

are specified, where $x_{i0}^0 \neq y_0^0$ and $x_{i1}^0 \neq y_1^0$. Here and below, unless otherwise stated, $i = 1, 2, \ldots, n$.

Definition 1. In game Γ, "soft" capture occurs if $T > 0$ and the measurable functions $u_i(t) = u_i(t, x_{i\alpha}^0, y_\alpha^0, \upsilon_t(\cdot)) \in V$ are such that, for any measurable function $\upsilon(\cdot)$, $\upsilon(t) \in V$, $t \in [0, T]$, time $\tau \in [0, T]$ and a number $q \in \{1, 2, \ldots, n\}$ exist such that

$$x_q(\tau) = y(\tau), \quad \dot{x}_q(\tau) = \dot{y}(\tau)$$

Instead of systems (1.1) and (1.2), we will examine the system

$$z_i^{(l)} + a_1 z_i^{(l-1)} + \ldots + a_l z_i = u_i - \upsilon, \quad u_i, \upsilon \in V \tag{1.3}$$

$$z_i(0) = z_{i0}^0 = x_{i0}^0 - y_0^0, \ldots, z_i^{(l-1)}(0) = z_{il-1}^0 = x_{il-1}^0 - y_{l-1}^0 \tag{1.4}$$

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We will denote by φ_q ($q = 0, 1, \dots, l-1$) the solutions of the equation

$$\omega^{(1)} + a_1\omega^{(l-1)} + \dots + a_l\omega = 0$$

with initial conditions

$$\omega(0) = 0, \dots, \omega^{(q-1)}(0) = 0, \quad \omega^{(q)}(0) = 1, \quad \omega^{(q+1)}(0) = 0, \dots, \omega^{l-1}(0) = 0$$

Assumption 1. All the roots of the characteristic equation

$$\lambda^l + a_1\lambda^{l-1} + \dots + a_l = 0 \tag{1.5}$$

are real and non-positive.

We will denote the roots of Eq. (1.5) by $\lambda_1, \dots, \lambda_s$ ($\lambda_1 < \lambda_2 < \dots < \lambda_s$), and their multiplicities respectively by $k_1, \dots, k_s$.

Lemma 1. Suppose Assumption 1 is satisfied and $\lambda_s = 0$. Then $\varphi_{l-1}(t) \ge 0, \dot{\varphi}_{l-1}(t) \ge 0$ for all $t \ge 0$.

Lemma 2. Suppose Assumption 1 is satisfied and $\lambda_s < 0$. Then
(1) $\varphi_{l-1}(t) \ge 0$ for all $t > 0$;
(2) a $\tau_0 > 0$ exists such that $\dot{\varphi}_{l-1}(t) > 0$ $t \in (0, \tau_0)$, $\dot{\varphi}_{l-1}(t) < 0$, $t \in (\tau_0, \infty)$.

The assertions of Lemmas 1 and 2 follow from a well-known result [7, p. 136].

Further, let

$$\xi_i(t) = \sum_{k=0}^{l-1} \varphi_k(t) z_{ik}^0$$

Then ξ_i can be represented in the form

$$\xi_i(t) = \sum_{j=1}^{l-1} e^{\lambda_j t} P_{ji}(t)$$

where P_{ji} are polynomials. We will assume that $\deg P_{si} = k_s - 1 = \gamma$ for all i, otherwise the pursuers initially achieve satisfaction of the given condition, selecting their controls $u_i(t)$ in a fairly small time segment such that the coefficients of t^γ in the polynomials P_{si} are non-zero.

We will consider the case

$$\lambda_s = 0, \quad k_s \ge 2 \tag{1.6}$$

and introduce the notation

$$M(t, \tau) = \min\left\{\frac{\varphi_{l-1}(t-\tau)}{t^\gamma}, \frac{\dot{\varphi}_{l-1}(t-\tau)}{\gamma t^{\gamma-1}}\right\}, \quad R(f, t, \tau) = \sum_{j=1}^{s-1} \frac{e^{\lambda_j(t-\tau)} f_j(t-\tau)}{\gamma(t-\tau)^{\gamma-1}}$$

Lemma 3. Suppose Assumption 1 and condition (1.6) are satisfied. Then, for any $T > 0$

$$\lim_{t\to\infty} \int_T^t M(t, \tau) d\tau = \infty$$

Proof. The functions $\varphi_{l-1}, \dot{\varphi}_{l-1}$ can be represented in the form

$$\varphi_{l-1}(t-\tau) = a_\gamma(t-\tau)^\gamma[1 + g_1(t-\tau)]$$

$$\dot{\varphi}_{l-1}(t-\tau) = a_\gamma\gamma(t-\tau)^{\gamma-1}[1 + g_2(t-\tau)]$$

where

$$g_1(t-\tau) = \sum_{l=0}^{\gamma-1} \frac{a_l}{(t-\tau)^{\gamma-l}} + R(P, t, \tau), \quad g_2(t-\tau) = \sum_{l=1}^{\gamma-1} \frac{b_l}{(t-\tau)^{\gamma-l}} + R(Q, t, \tau)$$

Suppose $\varepsilon \in (0, 1)$ and $\tau \in [0, \varepsilon t]$. Then $t - \tau \geq (1-\varepsilon)t$ and

$$|g_1(t-\tau)| \leq \sum_{r=1}^{\gamma} \frac{|a_{\gamma-r}|}{t^r(1-\varepsilon)^r} + \Sigma^1(t) = \Delta_1(t)$$

$$|g_2(t-\tau)| \leq \sum_{r=1}^{\gamma-1} \frac{|b_{\gamma-r}|}{t^r(1-\varepsilon)^r} + \Sigma^2(t) = \Delta_2(t)$$

where

$$\Sigma^k(t) = \sum_{j=1}^{s-1} e^{\lambda_j(1-\varepsilon)t} c_j^k(t), \quad c_j^1(t) = \frac{\max_{t \in [0, \varepsilon t]} |P_j(t-\tau)|}{t^\gamma(1-\varepsilon)^\gamma}, \quad c_j^2(t) = \frac{\max_{\tau \in [0, \varepsilon t]} |Q_j(t-\tau)|}{\gamma t^{\gamma-1}(1-\varepsilon)^{\gamma-1}}$$

Since $\Delta_1(t)$ and $\Delta_2(t) \to 0$ where $t \to \infty$, 9 time T_0 exists such that $|\Delta_1(t)| \leq 1/2$ and $|\Delta_2(t)| \leq 1/2$ for all $t > T_0$. Therefore

$$\varphi_{l-1}(t-\tau) \geq {}^1\!/_2\, a_\gamma (t-\tau)^\gamma, \quad \dot{\varphi}_{l-1}(t-\tau) \geq {}^1\!/_2\, a_\gamma \gamma (t-\tau)^{\gamma-1}$$

for all $t > T_0$ and $\tau \in [0, \varepsilon t]$.

Hence, for all (t, T) such that $T > T_0$ and $\varepsilon t > T$, the inequality

$$\int_T^t M(t,\tau)d\tau \geq \int_T^{\varepsilon t} M(t,\tau)d\tau \geq \int_T^{\varepsilon t} \frac{a_\gamma}{2} \frac{(t-\tau)^\gamma}{t^\gamma} d\tau \to \infty \quad \text{when} \quad t \to \infty$$

holds.

Further, let

$$z_i^0 = \lim_{t \to \infty} P_{si}(t)/t^\gamma, \quad \lambda(A, \upsilon) = \sup\{\lambda | \lambda \geq 0, -\lambda A \cap (V - \upsilon) \neq 0\}$$

$$\delta = \inf_{\upsilon \in V} \max_i \lambda(z_i^0, \upsilon) > 0$$

Assumption 2. The function $\lambda(z_i^0, \upsilon)$ are continuous at all points of the form (z_i^0, υ) for which $\lambda(z_i^0, \upsilon) > 0$.

Lemma 4. Suppose Assumptions 1 and 2 and condition (1.6) are satisfied and $\delta > 0$. Then a time T exists such that, for any admissible function υ, a number i will be found such that $h_i(T) \leq 0$, where

$$h_i(t) = 1 - \int_0^T \beta_i(T, \tau, \upsilon(\tau))d\tau$$

$$\beta_i(t, \tau, \upsilon) = \sup\{\lambda | \lambda \geq 0, -\lambda \mathcal{L}(\xi_i(t), t) \in \mathcal{L}(\varphi_{l-1}(t-\tau), t)(V - \upsilon)\}$$

$$\mathcal{L}(f(r), t) = \begin{Vmatrix} f(r)/t^\gamma \\ \dot{f}(r)/(\gamma t^{\gamma-1}) \end{Vmatrix}$$

Proof. Note that

$$\beta_i(t, \tau, \upsilon) = \min\left\{\frac{\varphi_{l-1}(t-\tau)}{t^\gamma}\lambda\left(\frac{\xi_i(t)}{t^\gamma}, \upsilon\right), \frac{\dot{\varphi}_{l-1}(t-\tau)}{\gamma t^{\gamma-1}}\lambda\left(\frac{\dot{\xi}_i(t)}{\gamma t^{\gamma-1}}, \upsilon\right)\right\}$$

Since

$$z_i^0 = \lim_{t \to \infty} \frac{\xi_i(t)}{t^\gamma} = \lim_{t \to \infty} \frac{\dot{\xi}_i(t)}{\gamma t^{\gamma-1}}$$

a moment T_0 exists such that

$$\max_i \lambda\left(\frac{\xi_i(t)}{t^\gamma}, \upsilon\right) \geq \frac{\delta}{2}, \quad \max_i \lambda\left(\frac{\dot{\xi}_i(t)}{\gamma t^{\gamma-1}}, \upsilon\right) \geq \frac{\delta}{2}$$

for all $t > T_0$ and $\upsilon \in V$. We will consider continuous functions h_i, taking into account that

$$h_i(0) = 1, \quad \sum_i h_i(T) \le n - \int_0^T \max_i \beta_i(T, \tau, \upsilon(\tau))d\tau$$

Suppose $T > T_0$. Then

$$\max_i \beta_i(T, \tau, \upsilon(\tau)) \ge \frac{\delta}{2} M(T, \tau)$$

and therefore

$$\int_0^T \max_i \beta_i(T, \tau, \upsilon(\tau))d\tau \ge \frac{\delta}{2}\int_{T_0}^T M(T, \tau)d\tau = g(T)$$

Consequently, $\Sigma_i h_i(T) \le n - g(T)$. Since $\lim_{T\to\infty} g(T) = +\infty$, a moment T_1 and a number i exist such that $h_i(T_1) \le 0$.

Let

$$\hat{T} = \inf\left\{T \ge 0: \inf_{\upsilon(\cdot) \in \Omega(T)} \max_i \int_0^T \beta_i(T, \tau, \upsilon(\tau))d\tau \ge 1\right\}$$

where $\Omega(T)$ is the set of all measurable functions υ defined in the segment $[0, T]$ and taking values in V.

By virtue of Lemma 4, $\hat{T} < \infty$.

Theorem 1. Suppose Assumptions 1 and 2 and condition (1.6) are satisfied and $\delta > 0$. Then, in game Γ, "soft" capture occurs.

Proof. Let $\upsilon : [0, \hat{T}] \to V$ be an arbitrary admissible control of the evader E and t_1 be the least positive root of the function h of the form

$$h(t) = 1 - \max_i \int_0^t \beta_i(\hat{T}, \tau, \upsilon(\tau))d\tau$$

Let $\hat{u}_i(\tau)$ be the lexicographic minimum among the solutions of the system

$$-\beta_i(\hat{T}, \tau, \upsilon(\tau))\mathcal{L}(\xi_i(\hat{T}), \hat{T}) = \mathcal{L}(\varphi_{l-1}(\hat{T} - \tau), \hat{T})(u - \upsilon(\tau))$$

We set the controls of the pursuers P_i, supposing that $u_i(\tau) = \hat{u}_i(\tau)$. We assume that $\beta_i(\hat{T}, \tau, \upsilon(\tau)) = 0$ when $\tau \in [t_1, \hat{T}]$. Then

$$\mathcal{L}(z_i(\hat{T}), \hat{T}) = \mathcal{L}(\xi_i(\hat{T}), \hat{T}) + \int_0^{\hat{T}} \mathcal{L}(\varphi_{l-1}(\hat{T} - \tau), \hat{T})(u_i(\tau) - \upsilon(\tau))d\tau =$$

$$= \mathcal{L}(\xi_i(\hat{T}), \hat{T})h_i(\hat{T}) = \mathcal{L}(\xi_i(\hat{T}), \hat{T})\left(1 - \int_0^{t_1} \beta_i(\hat{T}, \tau, \upsilon(\tau))d\tau\right) = 0$$

The theorem is proved.

We will consider the case

$$\lambda_s = 0, \quad k_s = 1 \tag{1.7}$$

and introduce the notations

$$M_1(t, \tau) = \min\left\{\varphi_{l-1}(t - \tau), \frac{\dot{\varphi}_{l-1}(t - \tau)e^{-\lambda_{s-1}t}}{t^{\mu}}\right\}, \quad \mathcal{L}_1(f(r), t) = \left\|\begin{matrix} f(r) \\ f(r)e^{-\lambda_{s-1}t}/t^{\mu}\end{matrix}\right\|$$

In this case

$$\dot{\varphi}_{l-1}(t) = \sum_{r=1}^{s-1} e^{\lambda_r t} Q_r(t), \quad \xi_i(t) = \sum_{j=1}^{s-1} e^{\lambda_j t} P_{ji}(t) + z_i^0, \quad \dot{\xi}_i(t) = \sum_{j=1}^{s-1} e^{\lambda_j t} Q_{ji}(t)$$

Let $\deg Q_{s-1}(t) = \mu$. We assume that $Q_{s-1i}(t) \neq 0$ and $\deg Q_{s-1i}(t) = \mu$ for all i. Let $z_i^1 = \lim_{t \to \infty} Q_{s-1i}/t^\mu$.

Lemma 5. Suppose Assumption 1 and condition (1.7) are satisfied. Then, for any $T > 0$

$$\lim_{t \to \infty} \int_T^t M_1(t, \tau) d\tau = \infty$$

Proof. The functions $\varphi_{l-1}, \dot{\varphi}_{l-1}$ can be represented in the form

$$\varphi_{l-1}(t-\tau) = a_\gamma + g_1(t-\tau)$$

$$\dot{\varphi}_{l-1}(t-\tau) = e^{\lambda_{s-1}(t-\tau)}(t-\tau)^\mu \lambda_{s-1} b_\mu [1 + g_2(t-\tau)]$$

where

$$g_1(t-\tau) = \sum_{j=1}^{s-1} e^{\lambda_j(t-\tau)} P_j(t-\tau)$$

$$g_2(t-\tau) = \sum_{j=1}^{s-2} \frac{e^{(\lambda_j - \lambda_{s-1})(t-\tau)} Q_j^1(t-\tau)}{(t-\tau)^\mu} + \sum_{r=0}^{\mu-1} \frac{b_r}{(t-\tau)^{\mu-r}}$$

Let $\varepsilon \in (0, 1)$ and $\tau \in [0, \varepsilon t]$. Then $t - \tau \geq (1-\varepsilon)t$. Therefore, for $g_1(t-\tau)$ and $g_2(t-\tau)$, the inequalities

$$g_j(t-\tau) \leq \Delta_j(t)$$

hold. Here, $\Delta_j(t) \to 0$ when $t \to \infty$. Consequently, a time T_0 exists such that

$$\varphi_{l-1}(t-\tau) \geq 1/2 a_\gamma, \quad \dot{\varphi}_{l-1}(t-\tau) \geq 1/2 e^{\lambda_{s-1}(t-\tau)}(t-\tau)^\mu b_\mu \lambda_{s-1}$$

for all $t > T_0$ and $\tau \in [0, \varepsilon t]$. Hence

$$\dot{\varphi}_{l-1}(t-\tau) e^{-\lambda_{s-1} t}/t^\mu \geq 1/2(1-\varepsilon)^\mu b_\mu \lambda_{s-1}$$

and therefore, for all $T > T_0$

$$\int_T^t M_1(t, \tau) d\tau \geq \int_T^{\varepsilon t} a d\tau \to \infty \quad \text{when} \quad t \to \infty$$

Further, let

$$\delta = \inf_{\upsilon \in V} \max_i \min\{\lambda(z_i^0, \upsilon), \lambda(z_i^1, \upsilon)\}$$

Assumption 3. The functions $\lambda(z_i^0, \upsilon)$ and $\lambda(z_i^1, \upsilon)$ are continuous at all points (z_i^0, υ) and (z_i^1, υ) such that $\lambda(z_i^0, \upsilon) > 0$ and $\lambda(z_i^1, \upsilon) > 0$.

Lemma 6. Suppose Assumptions 1 and 3 and condition (1.7) are satisfied and $\delta > 0$. Then, for any admissible function υ, a time T and a number i exist such that $h_i(T) \leq 0$, where

$$\beta_i(t, \tau, \upsilon) = \sup\{\lambda | \lambda \geq 0, -\lambda \mathcal{L}_1(\xi_i(t), t) \in \mathcal{L}_1(\varphi_{l-1}(t-\tau), t)(V - \upsilon)\}$$

Proof. From the condition $\delta > 0$ it follows that, for any $\upsilon \in V$, a number i exists such that $\lambda(z_i^0, \upsilon) > 0$ and $\lambda(z_i^1, \upsilon) > 0$. By virtue of Assumption 3 and the condition

$$z_i^0 = \lim_{t \to \infty} \xi_i(t), \quad z_i^1 = \lim_{t \to \infty} \dot{\xi}_i(t) e^{-\lambda_{s-1} t}/t^\mu$$

a time T_1 exists such that, for all $t > T_1$, the inequality

$$\inf_{\upsilon}\max_{i}\min\{\lambda(\xi_i(t), \upsilon), \lambda(\dot{\xi}_i(t)e^{-\lambda_{s-1}t}/t^{\mu}, \upsilon)\} \geq \frac{\delta}{2}$$

holds.

Let $T > T_1$. Then

$$\sum_i h_i(T) \leq n - \frac{\delta}{2}\int_{T_1}^{T} M_1(t, \tau)d\tau = n - g(T)$$

According to Lemma 5, $g(T) \to \infty$ when $T \to \infty$. Therefore, a time T_0 and a number i exist such that $h_i(T_0) \leq 0$.

Theorem 2. Suppose Assumptions 1 and 3 and condition (1.7) are satisfied and $\delta > 0$. Then, in game Γ, "soft" capture occurs.

Proof. Let $\upsilon : [0, \hat{T}] \to V$ be an arbitrary admissible control of the evader E and t_1 be the smallest positive root of the function h. Let $\hat{u}_i(\tau)$ be the lexicographic minimum among the solutions of the system

$$-\beta_i(\hat{T}, \tau, \upsilon(\tau))\mathcal{L}_1(\xi_i(\hat{T}), \hat{T}) = \mathcal{L}_1(\varphi_{l-1}(\hat{T} - \tau), \hat{T})(u - \upsilon(\tau))$$

We set the controls of the pursuers P_i, supposing that $u_i(\tau) = \hat{u}(\tau)$. We assume that $\beta_i(\hat{T}, \tau, \upsilon(\tau)) = 0$ when $\tau \in [t_1, \hat{T}]$. Then

$$\mathcal{L}_1(z_i(\hat{T}), \hat{T}) = \mathcal{L}_1(\xi_i(\hat{T}), \hat{T}) + \int_0^{\hat{T}} \mathcal{L}_1(\varphi_i(\hat{T} - \tau), \hat{T})(u_i(\tau) - \upsilon(\tau))d\tau =$$

$$= \mathcal{L}_1(\xi_i(\hat{T}), \hat{T})h_i(\hat{T}) = \mathcal{L}_1(\xi_i(\hat{T}), \hat{T})\left(1 - \int_0^{t_1} \beta_i(\hat{T}, \tau, \upsilon(\tau))d\tau\right) = 0$$

The theorem is proved.

Suppose

$$\mathcal{L}_2(f(r), t) = \frac{e^{-\lambda_s t}}{t^{\gamma}}\left\| \begin{matrix} f(r) \\ \dot{f}(r)/\lambda_s \end{matrix} \right\|$$

$$M_2(t, \tau) = \min\left\{\frac{\varphi_{l-1}(t-\tau)e^{-\lambda_s t}}{t^{\gamma}}, \frac{-\dot{\varphi}_{l-1}(t-\tau)e^{-\lambda_s t}}{t^{\gamma}}\right\}$$

Lemma 7. Suppose Assumption 1 is satisfied and $\lambda_s < 0$ and $\varepsilon \in (0, 1)$. Then, a time T_0 exists such that, for any $T > T_0$

$$\lim_{t \to \infty} \int_T^{\varepsilon t} M_2(t, \tau)d\tau = \infty$$

Proof. The functions $\varphi_{l-1}, -\dot{\varphi}_{l-1}$ can be represented in the form

$$\varphi_{l-1}(t-\tau) = a_{\gamma}(t-\tau)^{\gamma}e^{\lambda_s(t-\tau)}(1 + g_1(t, \tau))$$

$$-\dot{\varphi}_{l-1}(t-\tau) = a_{\gamma}(-\lambda_s)(t-\tau)^{\gamma}e^{\lambda_s(t-\tau)}(1 + g_2(t, \tau))$$

where

$$g_1(t, \tau) = \sum_{j=1}^{s-1} e^{(\lambda_j - \lambda_s)(t-\tau)}\frac{P_j(t-\tau)}{(t-\tau)^{\gamma}a_{\gamma}} + \sum_{l=0}^{\gamma-1}\frac{a_l}{(t-\tau)^{\gamma-l}}$$

$$g_2(t, \tau) = \sum_{j=1}^{s-1} e^{(\lambda_j - \lambda_s)(t-\tau)}\frac{Q_j(t-\tau)}{(t-\tau)^{\gamma}a_{\gamma}(-\lambda_s)} + \sum_{l=0}^{\gamma-1}\frac{b_l}{(t-\tau)^{\gamma-l}}$$

Let $\tau \in (0, \varepsilon t)$. Then $t - \tau \geq (1 - \varepsilon)t$, and therefore

$$|g_1(t, \tau)| \leq \Delta_1(t), \quad |g_2(t, \tau)| \leq \Delta_2(t)$$

and here $\Delta_1(t)$ and $\Delta_2(t) \to 0$ when $t \to \infty$.

Consequently, a time T_0 exists such that $|g_1(t, \tau)| \leq 1/2$ and $|g_2(t, \tau)| \leq 1/2$ for all $t > T_0$ and $\tau \in (0, \varepsilon t)$. Therefore

$$\frac{\varphi_{l-1}(t-\tau)e^{-\lambda_s t}}{t^\gamma} \geq \frac{a_\gamma (t-\tau)^\gamma e^{-\lambda_s t}}{t^\gamma}, \quad -\frac{\dot{\varphi}_{l-1}(t-\tau)e^{-\lambda_s t}}{t^\gamma} \geq \frac{a_\gamma (t-\tau)^\gamma e^{-\lambda_s t}(-\lambda_s)}{t^\gamma}$$

for all $t > T_0$ and $\tau \in (0, \varepsilon t)$.

Let $T > T_0$, $\varepsilon t > T$, $t(1 - \varepsilon) \geq \tau_0$ and $\tau \in (0, \varepsilon t)$. Then

$$\int_T^{\varepsilon t} M_2(t, \tau) d\tau \geq \int_T^{\varepsilon t} \frac{c(t-\tau)^\gamma e^{-\lambda_s t}}{t^\gamma} d\tau \to \infty \quad \text{when} \quad t \to \infty$$

Further, let

$$z_i^0 = \lim_{t \to \infty} \xi_i(t) \frac{e^{-\lambda_s t}}{t^\gamma}, \quad \delta = \inf_{v \in V} \max_i \lambda(z_i^0, v)$$

Note that

$$z_i^0 = \lim_{t \to \infty} \frac{\dot{\xi}_i(t) e^{-\lambda_s t}}{t^\gamma \lambda_s}$$

Lemma 8. Suppose Assumptions 1 and 2 are satisfied and $\lambda_s < 0$ and $\delta > 0$. Then a time T exists such that, for any admissible function v, a number i will be found such that $h_i(T) \leq 0$, where

$$\beta_i(T, \tau, v) = \begin{cases} \beta_i^1(T, \tau, v), & \text{if } T - \tau > \tau_0 \\ 0, & \text{if } T - \tau \leq \tau_0 \end{cases}$$

$$\beta_i^1(t, \tau, v) = \sup\{\lambda | \lambda \geq 0, -\lambda \mathcal{L}_2(\xi_i(t), t) \in \mathcal{L}_2(\varphi_{l-1}(t-\tau), t)(V - v)\}$$

(here τ_0 is the positive root of the function $\dot{\varphi}_{l-1}$).

The proof is similar to the proof of Lemma 6.

Theorem 3. Suppose Assumptions 1 and 2 are satisfied and $\lambda_s < 0$ and $\delta > 0$. Then, in game Γ, "soft" capture occurs.

The proof is similar to the proof of the corresponding theorems for $\lambda_s = 0$.

We will denote by int$X$, ri$X$ and co$X$ respectively the interior, relative interior and convex shell of the set $X \subset R^k$.

Example 1. Systems (1.3) and (1.4) have the form

$$\ddot{z}_i = u_i - v, \quad z_i(0) = z_i^0, \quad \dot{z}_i(0) = z_i^1; \quad \|u_i\| \leq 1, \quad \|v\| \leq 1$$

Then $\lambda_1 = 0$, $k_1 = 2$, $\varphi_0(t) = 1$ and $\varphi_1(1) = t$, and therefore

$$\xi_i(t) = z_i^0 + t z_i^1, \quad \dot{\xi}_i(t) = z_i^1$$

Assertion 1. Suppose $0 \in \operatorname{Intco}\{z_1^1, \ldots, z_n^1\}$. Then, in game Γ, "soft" capture occurs.

Example 2. Systems (1.3) and (1.4) have the form

$$z_i^{(3)} + 3\ddot{z}_i + 2\dot{z}_i = u_i - v, \quad \|u_i\| \leq 1, \quad \|v\| \leq 1$$

$$z_i(0) = z_{i0}^0, \quad \dot{z}_i(0) = z_{i1}^0, \quad \ddot{z}_i(0) = z_{i2}^0$$

Then

$$\lambda_1 = -2, \quad \lambda_2 = -1, \quad \lambda_3 = 0, \quad k_1 = k_2 = k_3 = 1$$

$$\varphi_0(t) = 1, \quad \varphi_1(t) = \frac{1}{2}e^{-2t} - 2e^{-t} + \frac{3}{2}, \quad \varphi_2(t) = \frac{1}{2}e^{-2t} - e^{-t} + \frac{1}{2}$$

and therefore

$$\xi_i(t) = e^{-2t}\left(\frac{1}{2}z_{i1}^0 + \frac{1}{2}z_{i2}^0\right) + e^{-t}(-2z_{i1}^0 - z_{i2}^0) + z_{i0}^0 + \frac{3}{2}z_{i1}^0 + \frac{1}{2}z_{i2}^0$$

We assume

$$z_i^0 = z_{i0}^0 + \frac{3}{2}z_{i1}^0 + \frac{1}{2}z_{i2}^0, \; z_i^1 = 2z_{i1}^0 + z_{i2}^0$$

We also assume that $z_i^0 \neq 0$ and $z_i^1 \neq 0$.

Assertion 2. Suppose

$$\min_{\upsilon}\max_{i}\min\{\lambda(z_i^0, \upsilon), \lambda(z_i^1, \upsilon)\} > 0$$

Then, in game Γ, "soft" capture occurs.

2. PURSUIT OF A GROUP OF EVADERS

Suppose the laws of motion of n pursuers $P_1, \dots, P_n$ with controls u_i and of m evaders $E_1, \dots, E_m$ with controls υ have the form

$$\ddot{x}_i = u_i, \quad \|u_i\| \le 1, \quad \ddot{y}_j = \upsilon, \quad \|\upsilon\| \le 1 \tag{2.1}$$

$$x_i(0) = x_i^0, \quad \dot{x}_i(0) = x_i^1, \quad y_j(0) = y_j^0, \quad \dot{y}_j(0) = y_j^1, \quad x_i^0 \neq y_j^0, \quad x_i^1 \neq y_j^1 \tag{2.2}$$

Note that all the evaders use the same control.

Definition 2. In game Γ, "soft" capture occurs if a time $T > 0$ and measurable functions

$$u_i(t) = u_i(t, x_{i\alpha}^0, y_\alpha^0, \upsilon_t(\cdot)), \quad \|u_i(t)\| \le 1$$

exist such that, for any measurable function $\upsilon(\cdot)$, $\|\upsilon(t)\| \le 1$, $t \in [0, T]$, a time $\tau \in [0, T]$ and numbers $q \in \{1, 2, \dots, n\}$ and $r \in \{1, 2, \dots, m\}$ exist such that

$$x_q(\tau) = y_r(\tau), \quad \dot{x}_q(\tau) = \dot{y}_r(\tau)$$

Instead of systems (2.1) and (2.2), we will examine the system

$$\ddot{z}_{ij} = u_i - \upsilon, \quad z_{ij}(0) = z_{ij}^0, \quad \dot{z}_{ij}(0) = z_{ij}^1 \tag{2.3}$$

We will assume that the initial data are such that
(a) for any set of indices $I \subset \{1, \dots, n\}$, $|I| \ge k + 1$, the condition

$$\operatorname{Int}\operatorname{co}\{x_i^1, i \in I\} \neq \varnothing$$

holds;
(b) any k vectors from the set $\{x_i^1 - y_j^1, y_s^1 - y_r^1, s \neq r\}$ are linearly independent.

Theorem 4. Suppose

$$\operatorname{Int}\operatorname{co}\{x_i^1\} \cap \operatorname{co}\{y_j^1\} \neq \varnothing \tag{2.4}$$

Then, in game Γ, "soft" capture occurs.

Proof. From the conditions of the theorem it follows that $n + m \ge k + 2$. By virtue of a well-known result ([8, Lemma 3]) the sets $I \subset \{1, \dots, n\}$ and $J \subset \{1, \dots, m\}$ exist such that

$$\text{rico}\{x_i^1, i \in I\} \cap \text{rico}\{y_j^1, j \in J\} \neq \varnothing$$

and $|I| + |J| = k + 2$. We will assume that

$$I = \{1, \ldots, q\}, \quad J = \{1, \ldots, l\}$$

where $q + l = k + 2$. From a well-known result [8, Lemma 2], the system $\{z_{ij}^1, i \in I, j \in J\}$ forms a positive basis. If $|J| = 1$, then "soft" capture follows from Assertion 1. We assume that $|J| \geq 2$. Let $c_\alpha^\beta = y_\alpha^1 - y_\beta^1$. Then $z_{i\alpha}^1 = z_{i1}^1 + c_1^\alpha$ for all $i \in I$, $\alpha \in J$ and $\alpha \neq 1$.

Therefore $\{z_{i1}^1, i \in I, c_1^\alpha, \alpha \in J, \alpha \neq 1)$ from a positive basis. Since $n \geq k + 1$, then $q + \alpha - 1 \in \{q + 1, \ldots, n\}$ for all $\alpha \in J$ and $\alpha \neq 1$. By a well-known result ([8, Lemma 1]), the system

$$\{z_{i1}^1, i \in I, z_{q+\alpha-11}^1 + \mu c_1^\alpha, \alpha \in J, \alpha \neq 1\}$$

forms a positive basis. By virtue of Lemma 4 a time $T > 0$ exists such that, for any admissible function $\upsilon(\cdot)$, a number i will be found such that

$$1 - \int_0^T \beta_{i1}(T, \tau, \upsilon(\tau))d\tau \leq 0$$

where

$$\mathcal{L}_3(f(r), t) = \left\| \begin{matrix} \dfrac{f(r)}{t} \\ \dot{f}(r) \end{matrix} \right\|$$

$$\beta_{j1}(t, \tau, \upsilon) = \sup\{\lambda | \lambda \geq 0, -\lambda\mathcal{L}_3(\xi_{j1}(t), t) \in \mathcal{L}_3((t-\tau), t)(V - \upsilon)\}$$

$$\xi_{i1}(t) = z_{i1}^0 + z_{i1}^1 t, \quad i \in I$$

$$\xi_{q+\alpha-11}(t) = z_{q+\alpha-11}^0 + (z_{q+\alpha-11}^1 + \mu c_1^\alpha)t, \quad \alpha \in J, \quad \alpha \neq 1, \quad V = \{\upsilon : \|\upsilon\| \leq 1\}$$

Suppose

$$T_0 = \inf\left\{ T \,|\, \inf_{\upsilon(\cdot)} \max_j \int_0^T \beta_{j1}(T, \tau, \upsilon(\tau))d\tau \geq 1 \right\}$$

$\upsilon(\cdot)$ is an arbitrary control of the evaders and t_1 is the smallest positive root of function h of the form

$$h(t) = 1 - \max_j \int_0^t \beta_{j1}(T_0, \tau, \upsilon(\tau))d\tau$$

Let $\hat{u}_j(\tau)$ be the lexicographic minimum among the solutions of the system

$$-\beta_{j1}(T_0, \tau, \upsilon(\tau))\mathcal{L}_3(\xi_{j1}(T_0), T_0) \in \mathcal{L}_3((T_0 - \tau), T_0)(u - \upsilon(\tau))$$

We specify the controls of the pursuers, assuming $u_i(t) = \hat{u}_i(t)$. We also assume that $\beta_{j1}(T_0, \tau, \upsilon(\tau)) = 0$ and $\tau \in [t_1, T_0]$. Then, from the system (2.3) we obtain

$$\begin{aligned} \dot{z}_{i1}(t) &= z_{i1}^1 h_i(t), \quad i \in I \\ \dot{z}_{q+\alpha-11}(t) + z_{q+\alpha-11}^1 + \mu c_1^\alpha &= (z_{q+\alpha-11}^1 + \mu c_1^\alpha) h_{q+\alpha-1}(t), \quad \alpha \in J, \quad \alpha \neq 1 \end{aligned} \tag{2.5}$$

From the definition of T_0 it follows that a value of r exists for which $h_r(T_0) = 0$. If $r \in I$, then, by Theorem 1, in game Γ, "soft" capture occurs. If $h_{q+\gamma-11}(T_0) = 0$ at a certain $\gamma \in J$, $\gamma \neq 1$, then $\dot{z}_{q+\gamma-11}(T_0) = -\mu c_1^\gamma$.

We will show that

$$\operatorname{rico}\{\dot{x}_i(T_0), i \in I\} \cap \operatorname{rico}\{\dot{y}_j(T_0), j \in J\} \neq \varnothing \tag{2.6}$$

Using equality (2.5) and the relation

$$\dot{z}_{i\alpha}(T_0) = \dot{z}_{i1}(T_0) + c_1^\alpha = \dot{z}_{i1}(T_0) + z_{i\alpha}^1 - z_{i1}^1$$

for all $\alpha \in J$, $\alpha \neq 1$, we obtain

$$z_{i\alpha}^1 = \dot{z}_{i\alpha}(T_0) - \dot{z}_{i1}(T_0) + z_{i1}^1 = \dot{z}_{i\alpha}(T_0) + H_{i1}^0 \dot{z}_{i1}(T_0), \quad H_{i1}^0 = (1 - h_{i1}(T_0))/h_{i1}(T_0)$$

According to the condition, the system $\{z_{ij}^1, i \in I, j \in J\}$ forms a positive basis, and therefore the system

$$\{\dot{z}_{i1}(T_0)/h_{i1}(T_0), \dot{z}_{i\alpha}(T_0) + H_{i1}^0 \dot{z}_{i1}(T_0), \alpha \in I \backslash \{1\}\}$$

forms a positive basis. Since $h_{i1}(T_0) \in (0, 1]$, the system $\{\dot{z}_{ij}(T_0), i \in I, j \in J\}$ forms a positive basis. Hence, using a well-known result [8, Lemma 2], we obtain relation (2.6). Since $\dot{z}_{q+\alpha_0-11}(T_0) = -\mu c_1^{\alpha_0}$ and condition (2.6) is satisfied, then, using a well-known result [8, Lemma 4], we obtain

$$\operatorname{rico}\{\dot{x}_i(T_0), i \in I, \dot{x}_{q+\alpha_0-11}(T_0)\} \cap \operatorname{rico}\{\dot{y}_j(T_0), j \in 1, j \in J\} \neq \varnothing$$

Assume $\alpha_0 = 2$. Further, we suppose that

$$I = \{1, 2, \ldots, q+1\}, \quad j = \{2, \ldots, l\}$$

For the sets I and J, the condition (2.4) holds, and here the number of evaders participating in the given condition has been reduced by one. Taking T_0 as the initial time, we repeat our reasoning until the number of evaders becomes equal to one. We will have

$$\operatorname{rico}\{\dot{x}_i(\tau), i \in I\} \cap \operatorname{rico}\{\dot{y}_j(\tau), j \in J\} \neq \varnothing$$

at a certain instant $\tau > 0$, and here $|I| = k + 1$ and $|J| = 1$. Now, capture follows from Assertion 1. The theorem is proved.

Theorem 5. Suppose

$$\operatorname{Int}\operatorname{co}\{x_i^1\} \cap \operatorname{co}\{y_j^1\} \neq \varnothing$$

Then, in game Γ, digression from "soft" capture occurs.
The proof follows from a well-known result [9].

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